

Groups with involution, and quasigroups with cracovian representations

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1

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Abstract In groups with involution a nonassociative product of elements is defined, which leads to the definition of a certain type of quasigroups. These quasigroups are represented by square tables of complex numbers, with inverses, which differ from the matrix representations of groups in the rule of performing the product of two tables. The row-by-column product of two matrices in representations of groups is replaced by the column-by-column product, which is called the cracovian product, in representations of the defined type of quasigroups. The matrices undergoing the column-by-column product are called cracovians. The basic properties of the quasigroups connected with groups with involution are determined while only a summary of the properties of cracovian algebra is presented, as the basis of cracovian representation theory for the quasigroups connected with groups with involution. Clifford groups are groups with involution and the quasigroups connected with them are determined. The orthogonal and pseudo-orthogonal rotation groups belong to groups with involution. An analogy is drawn between Weyl's "hidden" symmetry group of an object, and the quasigroup connected with the group with involution.

Keywords Groups with involution, quasigroups, Clifford groups, cracovian representations

1 Introduction

The work in the field of quasigroups acquired impetus with the investigations presented in the "Projektive Ebenen" by G. Pickert (1955)[29], in the "Binary Systems" by R. H. Bruck (1958 and 1966)[14], and in the "Foundations of Quasigroups and Loops" by V. D. Belousov (1967) [12]. It was followed by the books which appeared in the last decade of the twentieth century: "Quasigroups and Loops: Introduction" by H. O. Pflugfelder (1990)[28], "Quasigroups and Loops: Theory and Application" edited by O. Chein, H. O. Pflugfelder and J. D. H. Smith (1990) [15], and "Smooth Quasigroups and Loops" by L. V. Sabinin (1999)[30]. A survey of papers on nonassociative geometry with the reference to space-time was presented by L. V. Sabinin [31]. The line of thought of that survey was pursued in [33], in an application of nonassociative geometry to special relativity. Quasigroups connected with groups with involution, and their cracovian representations were introduced in [21, 22, 23]. The attempts of applying nonassociative algebras in quantum mechanics, were initiated by Jordan [18], and Jordan *et al.*, [19, 20]. Full expositions of Jordan algebras were given by Braun and Koecher [13] and by Jacobson [17]. The nonassociative and noncommutative column-by-column product, (or, alternatively, row-by-row product) of two tables of symbols or complex numbers, was defined and applied in theoretical astronomy by Banachiewicz in the year 1924, at the Jagellonian University in Cracow. The name "cracovian" was coined in connection with that. The cracovian algebra was developed by Banachiewicz in a series of

papers over the span of thirty years, of which we quote the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. It was presented in the books by Banachiewicz [11], by Sierpiński [34], by Łukaszewicz and Warmus [26], and by Kociński [23]. In the last book a list of references concerning cracovian algebra and some of its applications are given.

2 The quasigroups connected with groups with involution

We consider a finite group G of order N , with elements g and unit element e , for which an involution operation $I(g)$ exists with the following properties:

$$\begin{aligned} I(g) &\in G, \quad \forall g \in G \\ I[I(g)] &= g \\ I(g_1 g_2 \dots g_p) &= I(g_p) I(g_{p-1}) \dots I(g_2) I(g_1), \quad \forall g_1, g_2, \dots, g_p \in G \\ I(\pm e) &= \pm e \end{aligned} \tag{2.1}$$

In the following we will denote the group elements by g_1, g_2, \dots, g_N or, alternatively, by a, b, c, \dots, z .

Definition 2.1. We define the following group automorphism by the equality

$$I(a)b \stackrel{\text{def}}{=} c, \quad \text{for a fixed } a, \quad \forall a, b, c \in G \tag{2.2}$$

where, for brevity, a, b and c now denote arbitrary products of the elements g_1, g_2, \dots, g_N .

Corollary 2.1. The automorphisms defined in Eq. (2.2) together with Eq. (2.1) form groups.

Proof. The product of two such automorphisms yields another automorphism of the same type,

$$I(d)[I(a)b] = I(d)I(a)b = I(ad)b$$

The product of three automorphisms is associative since we have

$$[I(c)I(b)]I(a)d = I(bc)I(a)d = I(abc)d$$

$$I(c)[I(b)I(a)d] = I(c)I(ab)d = I(abc)d$$

There exists the unit element since $I(1)a = a = aI(1)$, and for each automorphism $I(a)$, there exists the inverse automorphism $I(a^{-1})$ since we have

$$I(a^{-1})[I(a)b] = I(aa^{-1})b = b$$

$$I(a)[I(a^{-1})b] = I(a^{-1}a)b = b$$

Definition 2.2. The "dot" product denoted by (\cdot) is defined by

$$a \cdot b \stackrel{\text{def}}{=} I(b)a, \quad \forall a, b \in G \quad (2.3)$$

where on the right hand side we are dealing with the associative product in the group G .

Observation 2.1. To distinguish the elements of the group G undergoing the associative product from the same elements in the "dot" product, we will write the elements of the group G in the "dot" product with a caret. This means that the product $a \cdot b$ will be rewritten as $\hat{a} \cdot \hat{b}$.

Definition 2.3. If e denotes the unit element in the group G in Eq. (2.1), and a is an element in that group, the right unit element τ for the "dot" product in Eq.(2.3), is defined by the equality

$$\hat{a} \cdot \tau = I(e)a = a \rightarrow \hat{a}, \quad \forall a \in G \quad (2.4)$$

where the arrow on the right hand side indicates the passage from the group element a to the same element undergoing the "dot" product, then denoted by \hat{a} .

Corollary 2.2. The right unit element τ is not at the same time a left unit element, since from Eq. (2.3) we obtain

$$\tau \cdot \hat{a} = I(a)e = I(a) \quad (2.5)$$

and, in a general case, we have $I(a) \neq a$.

Corollary 2.3. The right unit element τ applied from the left twice, does not change any element in the "dot" product, since we have

$$\tau \cdot (\tau \cdot \hat{a}) = I[I(a)e]e = a \rightarrow \hat{a}, \quad \forall a \in G \quad (2.6)$$

Definition 2.4. The product defined in Eq. (2.3) is performed from left to right:

$$\hat{a} \cdot \hat{b} \cdot \hat{c} \cdot \dots \cdot \hat{z} = \{[(\hat{a} \cdot \hat{b}) \cdot \hat{c}] \cdot \dots \cdot \hat{z}\} \quad (2.7)$$

Corollary 2.4. From Eq. (2.3) it follows that

$$(\hat{a} \cdot \hat{b}) \cdot \hat{c} = \hat{a} \cdot [\hat{c} \cdot (\hat{\tau} \cdot b)], \quad \forall a, b, c \in G \quad (2.8)$$

Proof. We have:

$$(\hat{a} \cdot \hat{b}) \cdot \hat{c} = I(c)(\hat{a} \cdot \hat{b}) = I(c)I(b)a$$

and

$$\hat{a} \cdot [\hat{c} \cdot (\hat{\tau} \cdot \hat{b})] = I[I(\hat{\tau} \cdot \hat{b})c]a = I(c)I(b)a$$

The "dot" product is nonassociative.

Corollary 2.5. From Eqs. (2.3) and (2.5) it follows that

$$ab = \hat{b} \cdot I(a), \quad \forall a, b \in G \quad (2.9)$$

where on the right hand side we have $I(a) = a' \rightarrow \hat{a}'$.

Definition 2.5. If a^{-1} is the inverse of the element a in the group G , then the right inverse of \hat{a} is defined by

$$\hat{a}^{-1} = I(a^{-1}) \quad (2.10)$$

since from Eq. (2.10) we then obtain: $e = a^{-1}a = \hat{a} \cdot I(a^{-1}) \rightarrow \tau$.

Corollary 2.6. The right inverse in Eq. (2.10) at the same time is the left inverse, since from Eqs. (2.7) and (2.10) we obtain

$$\tau = \tau \cdot \tau = \tau \cdot (\hat{a} \cdot \hat{a}^{-1}) = [\tau \cdot (\tau \cdot \hat{a}^{-1})] \cdot \hat{a} = \hat{a}^{-1} \cdot \hat{a} \quad (2.11)$$

Corollary 2.7. The right or left inverse of τ is equal to τ .

Corollary 2.8. For the "dot" product there is no left unit element.

Proof. If \hat{x} were a left unit element, we would have $\hat{x} \cdot (\hat{b} \cdot \hat{c}) = \hat{b} \cdot \hat{c}$. However, from Eq. (2.6) at the same time we would have $\hat{x} \cdot (\hat{b} \cdot \hat{c}) = [\hat{x} \cdot (\tau \cdot \hat{c})] \cdot \hat{b} = (\tau \cdot \hat{c}) \cdot \hat{b}$. We therefore obtain: $\hat{b} \cdot \hat{c} = (\tau \cdot \hat{c}) \cdot \hat{b}$, which, in a general case, is not true.

Corollary 2.9. From Eqs. (2.2), (2.3), (2.6) (2.7) and (2.9) it follows that the elements of the group G in Eq. (2.1) undergoing the "dot" product, form a quasigroup QG with the right unit element. The Cayley table of this quasigroup is determined on the basis of the product definition in Eq. (2.3) and the Cayley table of the group G .

Observation 2.2. If the "dot" product definition in Eq.(2.2) were replaced by the definition

$$\hat{a} \cdot \hat{b} \stackrel{\text{def}}{=} aI(b), \quad \forall a, b \in G \quad (2.12)$$

we again would obtain Eqs. (2.3), (2.6), (2.7) and (2.9), which define a quasigroup QG , connected with the group G . The choice between the two definitions of the "dot" product will be dictated by the homomorphism existing between the quasigroup based on Eq. (2.3), and square cracovians with inverses, and the lack of such a homomorphism when the product definition in Eq. (2.12) is accepted.

Observation 2.3. In the following formulas the definition in Eq. (2.3) which relates the quasigroup "dot" product with the group product will not be used. To simplify notation, we no longer will write the "caret" above the elements of the group G undergoing the quasigroup product.

Observation 2.4. In order to avoid unnecessary brackets in the formulas, we will omit

from now on the "dot" while multiplying any element of the quasigroup from the left or from the right by the right unit element τ . Consequently, instead of $\tau \cdot a$ or $a \cdot \tau$, we will write τa or $a\tau$, respectively; instead of $b \cdot (\tau \cdot a)$ we will write $b \cdot \tau a$, and $a \cdot \tau \cdot b$ will be replaced by $a\tau \cdot b$. In particular, we now will rewrite Eq. (2.7) in the form

$$(a \cdot b) \cdot c = a \cdot (c \cdot \tau b) \quad (2.13)$$

Corollary 2.10. From Eqs. (2.13) and (2.6) we find that

$$\tau(a \cdot b) = b \cdot a \quad (2.14)$$

Corollary 2.11. From $a \cdot a^{-1} = \tau$, it follows that $(\tau a) \cdot (\tau a^{-1}) = \tau$.

Proof. Writing $a^{-1} = b$, and multiplying $a \cdot b = \tau$ from the right by $(\tau b)^{-1}$ we obtain $(a \cdot b) \cdot (\tau b)^{-1} = a \cdot [(\tau b)^{-1} \cdot \tau b] = \tau(\tau b)^{-1} = a$, since $a \cdot b = \tau$. Multiplying the last equality from the left by τb we obtain $(\tau b)^{-1} = \tau a$, and multiplying both sides of the last equality by τb from the right we obtain,

$$(\tau b)^{-1} \cdot (\tau b) = (\tau a) \cdot (\tau b) = (\tau a) \cdot (\tau a^{-1}) = \tau \quad (2.15)$$

which proves the statement.

Corollary 2.12. We have

$$(a \cdot b)^{-1} = a^{-1} \cdot b^{-1} \quad (2.16)$$

Proof. From Eqs. (2.13) and (2.6) we obtain

$$\begin{aligned} (a \cdot b) \cdot (a^{-1} \cdot b^{-1}) &= a \cdot [(a^{-1} \cdot b^{-1}) \cdot \tau b] = \\ &= a \cdot [a^{-1} \cdot (\tau b \cdot \tau b^{-1})] = a \cdot a^{-1} = \tau \end{aligned}$$

which proves the statement.

Corollary 2.13. From Eq. (2.16) we obtain

$$(\tau a)^{-1} = \tau a^{-1} \quad (2.17)$$

since $\tau^{-1} = \tau$.

Corollary 2.14. For any finite number of factors g_1, g_2, \dots, g_p , we have,

$$(g_1 \cdot g_2 \cdot \dots \cdot g_p)^{-1} = g_1^{-1} \cdot g_2^{-1} \cdot \dots \cdot g_p^{-1} \quad (2.18)$$

since if this equality is valid for k factors, then owing to Eq. (2.16) it also is valid for $(k+1)$ factors,

$$[(g_1 \cdot g_2 \cdot \dots \cdot g_k) \cdot g_{(k+1)}]^{-1} = (g_1 \cdot g_2 \cdot \dots \cdot g_k)^{-1} \cdot g_{(k+1)}^{-1} \\ g_1^{-1} \cdot g_2^{-1} \cdot \dots \cdot g^{-1} \cdot g_{(k+1)}^{-1}$$

Corollary 2.15. From Eqs. (2.13) and (2.14) we find that

$$\tau(a \cdot b \cdot c) = c \cdot \tau b \cdot a \quad (2.19)$$

since from Eq. (2.14) we have $\tau[(a \cdot b) \cdot c] = c \cdot (a \cdot b)$, and from Eq. (2.17) we obtain $c \cdot (a \cdot b) = c \cdot \tau b \cdot a$.

Corollary 2.16. For any finite number of p factors we have

$$\tau(g_1 \cdot g_2 \cdot \dots \cdot g_{(p-1)} \cdot g_p) = g_p \cdot \tau g_{(p-1)} \cdot \dots \cdot \tau g_2 \cdot g_1 \quad (2.20)$$

Proof. We write: $g_1 \cdot g_2 \cdot \dots \cdot g_p = (g_1 \cdot g_2 \cdot \dots \cdot g_{(p-2)}) \cdot g_{(p-1)} \cdot g_p = W$, and then owing to Eqs. (2.19) and (2.13) we obtain the equalities

$$\tau W = g_p \cdot \tau g_{(p-1)} \cdot (g_1 \cdot g_2 \cdot \dots \cdot g_{(p-2)}) = (g_p \cdot \tau g_{(p-1)}) \cdot [(g_1 \cdot g_2 \cdot \dots \cdot g_{(p-3)}) \cdot g_{(p-2)}] = \\ (g_p \cdot \tau g_{(p-1)} \cdot \tau g_{(p-2)}) \cdot [(g_1 \cdot g_2 \cdot \dots \cdot g_{(p-4)}) \cdot g_{(p-3)}] = \dots = g_p \cdot \tau g_{p-1} \cdot \dots \cdot \tau g_2 \cdot g_1$$

Clifford quasigroups [21, 22, 23]. As an example of quasigroups connected with groups with involution we will consider the quasigroups connected with Clifford groups. We consider Clifford algebras with the generators $\gamma_1, \gamma_2, \dots, \gamma_N$, $N = 1, 2, \dots$, and with the structural condition:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}, \quad \mu, \nu = 1, 2, \dots, N \quad (2.21)$$

For a fixed N , the basis of a Clifford algebra consists of the unit element 1, the generators $\gamma_1, \dots, \gamma_N$, and all linearly independent products of these generators. The dimension of this algebra is 2^N . Since the respective Clifford group G contains the element (-1) , the order of the group G is equal to 2^{N+1} .

Definition 2.6. The involution operation in a Clifford group G is defined by

$$I(\gamma_\mu) \stackrel{\text{def}}{=} -\gamma_\mu, \quad I(I(\gamma_\mu)) \stackrel{\text{def}}{=} \gamma_\mu, \quad I(\pm 1) \stackrel{\text{def}}{=} \pm 1 \\ I(\gamma_\mu \gamma_\nu \dots \gamma_\sigma) \stackrel{\text{def}}{=} I(\gamma_\sigma) \dots I(\gamma_\nu) I(\gamma_\mu), \quad \gamma_\mu, \gamma_\nu, \dots, \gamma_\sigma \in G \quad (2.22)$$

For the group elements consisting of products of γ 's, we define the notation

$$\gamma_\mu \gamma_\nu \dots \gamma_\sigma \stackrel{\text{def}}{=} \gamma_{\mu\nu\dots\sigma} \quad (2.23)$$

Definition 2.7. We define the following group automorphism by the equality:

$$I(\gamma_A)\gamma_B \stackrel{\text{def}}{=} \gamma_C, \quad \text{for a fixed } \gamma_A \quad (2.24)$$

where, for brevity, γ_A, γ_B , and γ_C now denote arbitrary elements of a Clifford group, i.e., also arbitrary products of the elements γ_α , $\alpha = 1, 2, \dots, N$. The inverse of any $\gamma_A = \gamma_\alpha \gamma_\beta \dots \gamma_\rho$ is equal to $\gamma_A^{-1} = \gamma_\rho \dots \gamma_\beta \gamma_\alpha$.

Corollary 2.17. The automorphisms defined in Eq. (2.24) together with Eq. (2.22) form groups.

Corollary 2.18. The automorphism defined in Eq. (2.25) preserves the structural condition in Eq. (2.21).

Definition 2.8. We define the product

$$(\hat{\gamma}_{\mu\nu\dots\sigma}) \cdot (\hat{\gamma}_{\epsilon\eta\dots\rho}) \stackrel{\text{def}}{=} I(\gamma_{\epsilon\eta\dots\rho})\gamma_{\mu\nu\dots\sigma} \quad (2.25)$$

where the product on the right hand side is the associative product in Clifford groups, and where the γ -symbols in the "dot" product are distinguished with a "caret" from the same γ -symbols in the associative product. For single- γ group elements this definition reads

$$\hat{\gamma}_\mu \cdot \hat{\gamma}_\nu \stackrel{\text{def}}{=} I(\gamma_\nu)\gamma_\mu \quad (2.26)$$

Eqs. (2.4) through (2.26) determine Clifford quasigroups connected with Clifford groups.

Observation 2.14. A respective nonassociative Clifford algebra has a basis consisting of the right unit element τ , the generators $\hat{\gamma}_\mu$, $\mu = 1, \dots, N$, fulfilling the condition

$$\hat{\gamma}_\mu \cdot \hat{\gamma}_\nu + \hat{\gamma}_\nu \cdot \hat{\gamma}_\mu = -2\delta_{\mu\nu}\tau, \quad \mu, \nu = 1, \dots, N \quad (2.27)$$

and of all linearly independent products of these generators. The dimension of this algebra is 2^N . To perform the multiplication of these generators or of their products, we have to apply Eqs. (2.26) or (2.25). Since the respective Clifford quasigroup contains the element $-\tau$, the order of that quasigroup is equal to 2^{N+1} .

3 The Basic Properties of Cracovian Algebra

The proofs of the basic properties of cracovian algebra can be found in [23].

Observation 3.1. A square or a rectangular table of symbols or complex numbers can be called a matrix or a cracovian, depending on the definition of the product of two such tables. The definitions of a product with a scalar, of a sum or a difference of two matrices, of the equality of two matrices, of a symmetric or an antisymmetric matrix, and of a transposed

matrix, therefore carry over from matrices on cracovians.

Definition 3.1. A rectangular cracovian A is defined as the table of $m \times n$ elements

$$\left\{ \begin{array}{cccc} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{array} \right\} \quad (3.1)$$

where the first index (k) of an element a_{kl} , $k = 1, 2, \dots, m$, denotes a column and the second index (l), $l = 1, 2, \dots, n$, denotes a row, and where wavy brackets are used to distinguish a cracovian table of elements from a matrix table of the same elements, for which ordinary brackets are used.

Definition 3.2. The product of two cracovians A and B , denoted by $A \cdot B$ is obtained by multiplying the columns of A by the columns of B . This type of product can be performed only when the two cracovians have the same number of rows. The element in the k -th column and in the l -th row of the cracovian $A \cdot B$ is obtained by multiplying the k -th column of A by the l -th column of B , hence

$$(A \cdot B)_{kl} = \sum_i a_{ki} b_{li} \quad (3.2)$$

where the summation over the index i extends over all rows.

Observation 3.2. It will be shown that the cracovian product in Eq. (3.2) leads to the definition of cracovian quasigroups, having the properties of the quasigroups defined in Section 2. This justifies the use of the dot in the definition in Eq. (3.2).

Observation 3.3. The cracovian product can also be defined in the terms of row-by-row multiplication [23].

Corollary 3.1. From Eq. (3.2) it follows that the multiplication of cracovians is noncommutative.

Definition 3.3. The square cracovian T which is called the transposing cracovian is defined by its elements t_{kj} ,

$$t_{kj} = 1, \quad \text{if } k = j, \quad \text{and} \quad t_{kj} = 0, \quad \text{if } k \neq j, \quad k, j = 1, \dots, n \quad (3.3)$$

The number of rows n of the square cracovian T , is equal to the number of rows of the cracovian which is multiplied by T from the left or from the right.

Corollary 3.2. Any cracovian A multiplied by the transposing cracovian T from the right remains unchanged, and it is changed to the transposed cracovian after the multiplication by T from the left, since we have:

$$(A \cdot T)_{kl} = \sum_i a_{ki} t_{li} = a_{kl} \quad (3.4)$$

$$(T \cdot A)_{kl} = \sum_i t_{ki} a_{li} = a_{lk} \quad (3.5)$$

Corollary 3.3. The transposing cracovian T is the right unit cracovian, however, it is not a left unit cracovian.

Corollary 3.4. We have:

$$T \cdot T = T \quad (3.6)$$

Corollary 3.5. The cracovian T is the only cracovian having the properties specified in Eqs. (3.4), (3.5) and (3.6).

Definition 3.5. The multiplication of cracovians is performed from left to right:

$$A \cdot B \cdot C \cdot D \dots Z := \{[(A \cdot B) \cdot C] \cdot D \dots Z\} \quad (3.7)$$

This multiplication is possible when the number of rows of a particular factor in the chain of factors is equal to the number of rows of the preceding resultant cracovian.

Corollary 3.6. For any cracovian A we have the equality:

$$T \cdot (T \cdot A) = A \quad (3.8)$$

Corollary 3.7. For any two cracovians A and B with the same number of rows we have the equality:

$$T \cdot (A \cdot B) = (B \cdot A) \quad (3.9)$$

Proof. According to Eq. (3.5), the element (kl) of $A \cdot B$ turns into the element (lk) of $T \cdot (A \cdot B)$, and the latter is equal to the element (lk) of $B \cdot A$.

Corollary 3.8. The square $A^2 = A \cdot A$, of any nonzero cracovian A is a symmetric cracovian, since

$$T \cdot (A \cdot A) = A \cdot A \quad (3.10)$$

Definition 3.5. In order to avoid unnecessary brackets in the formulas, from now on we will omit the dot (\cdot) in the product of any cracovian with the transposing cracovian T , from the left side or from the right side. This means that we will write TA instead of $T \cdot A$, and AT instead of $A \cdot T$. Further on we will write $B \cdot TA$ instead of $B \cdot (T \cdot A)$, and $AT \cdot B$ will replace $(A \cdot T) \cdot B$.

Lemma 3.1. [23]. For three cracovians A, B and C we have the equality:

$$(A \cdot B) \cdot C = A \cdot (C \cdot TB) \quad (3.11)$$

Corollary 3.9. From Eqs. (3.9) and (3.11), it follows that

$$T[(A \cdot B) \cdot C] = (C \cdot TB) \cdot A \quad (3.12)$$

Corollary 3.10. For an arbitrary finite number of factors we obtain,

$$T(A_1 \cdot A_2 \cdot \dots \cdot A_{k-1} \cdot A_k) = A_k \cdot TA_{k-1} \cdot \dots \cdot TA_2 \cdot A_1 \quad (3.13)$$

Corollary 3.11. For the column-by-column product there is no cracovian with the property of the left unit cracovian.

Proof. If X were the left unit cracovian, we would have: $X \cdot (B \cdot C) = B \cdot C$. From Eq. (3.11) at the same time we obtain: $X \cdot (B \cdot C) = [X \cdot (TC)] \cdot B = TC \cdot B$. We therefore obtain the equality: $B \cdot C = TC \cdot B$, which in a general case is not true.

Corollary 3.12. The relations between the matrix product and the cracovian product of two tables A and B , and of three tables A , B and C , are given by the respective equalities:

$$AB = B \cdot TA, \quad \text{and} \quad ABC = C \cdot (TB) \cdot (TA) \quad (3.14)$$

$$\tilde{B}A = A \cdot B, \quad \text{and} \quad \tilde{C}\tilde{B}A = A \cdot B \cdot C \quad (3.15)$$

where on the left hand side of these equalities, the tables undergo the matrix product, and on the right hand side the same tables undergo the cracovian product, and where the symbol \sim denotes the transposed matrix [21, 23].

Definition 3.6. The right inverse of a square cracovian A is defined as the cracovian A^{-1} for which we have:

$$A \cdot A^{-1} = T \quad (3.16)$$

From Eqs. (3.8) and (3.11) it follows that A^{-1} is also the left inverse.

Corollary 3.13. For a square cracovian A with the inverse A^{-1} , we have

$$(TA) \cdot (TA^{-1}) = T \quad (3.17)$$

Corollary 3.14. For the product of two square cracovians A_1 and A_2 , having the inverses we have:

$$(A_1 \cdot A_2)^{-1} = A_1^{-1} \cdot A_2^{-1} \quad (3.18)$$

Corollary 3.15. From Eq. (3.18) we obtain:

$$(TA)^{-1} = TA^{-1} \quad (3.19)$$

Observation 3.4. A square table of symbols or complex numbers has the cracovian inverse if and only if it has the matrix inverse.

Lemma 3.2 [23]. Between the matrix inverse A_m^{-1} of a square table of symbols or of complex numbers, and the cracovian inverse A_c^{-1} of that table holds the relation:

$$A_m^{-1} = T(A_c^{-1}) \quad (3.20)$$

The matrix inverse of a square table is equal to the transposed cracovian inverse of that table.

In the following formulas we start with the matrix expressions, and employing Eqs. (3.15) and (3.20), we determine the respective cracovian expressions.

Corollary 3.16. Let \vec{e}_m and \vec{e}_c denote the one-column matrix and the one-column cracovian, respectively, constructed from the basis vectors $\vec{e}_1, \dots, \vec{e}_n$, of an n -dimensional linear vector space. Let S_m and S_c be the respective matrix and cracovian transformation of these basis vectors. The change of basis then is defined by the expressions:

$$\vec{e}'_m = \tilde{S}_m \vec{e}_m = \vec{e}_c \cdot T(TS_c) = \vec{e}_c \cdot S_c = \vec{e}'_c \quad (3.21)$$

where \tilde{S}_m denotes the transposed matrix, and where in the first equality we are dealing with the matrix product, while in the next two equalities we are dealing with the cracovian product.

Corollary 3.17. Let x_m and x_c (or y_m and y_c) denote the one-column matrix (or the one-column cracovian) constructed from the components x_1, x_2, \dots, x_n (or y_1, y_2, \dots, y_n), of a vector, which is referred to the two bases in Eq. (3.21), respectively. The relation between the two sets of components is given by

$$x_m = S_m y_m = y_c \cdot TS_c = x_c \quad (3.22)$$

where the transformation table S , has been identified with the matrix S_m or with the cracovian S_c , depending on the type of the employed product.

Corollary 3.18. Let x_m and x_c denote the column matrix and column cracovian, respectively, constructed from the components (x_1, x_2, \dots, x_n) of a vector in an n -dimensional linear vector space. A linear mapping of that vector, expressed in the terms of matrices or cracovians is given by

$$x'_m = A_m x_m = x_c \cdot T A_c = x'_c \quad (3.23)$$

where the table A is identified with a matrix A_m or with a cracovian A_c , depending on the type of the employed product.

Corollary 3.19. The relation between matrix and cracovian linear transformations A_m and B_m , or A_c and B_c , respectively, which determine the same linear mapping referred to two bases which are connected according to Eq. (3.21) is given by

$$B_m = S_m^{-1} A_m S_m = S_c \cdot T A_c \cdot S_c^{-1} = B_c \quad (3.24)$$

where the second equality connects an expression in matrix product with the same expression in cracovian product, and that connection follows from the second of Eqs. (3.15) and from Eq. (3.20).

Corollary 3.20. From the commutation condition of two matrix tables A_m and B_m : $A_m B_m = B_m A_m$, and the relation between matrix product and cracovian product of two tables A and B in the first of equations in Eq. (3.15), we obtain the commutativity condition of two cracovians A_c and B_c in the form:

$$A_c \cdot T B_c = B \cdot T A_c \quad (3.25)$$

4 Cracovian representations of quasigroups connected with groups with involution

Observation 4.1. To the definition of the "dot" product in Eq. (2.3) corresponds the relation between cracovian and matrix product in the first of the two relations in Eq. (3.15). The consequences following from the product in Eq. (2.3) are deduced in Eqs. (2.4) through (2.20). The consequences which follow from the cracovian product defined in Eq. (3.2) are deduced in Eqs. (3.3) through (3.20); the first of the two relations between cracovian product and matrix product in Eq. (3.15), appears among them. To each consequence of the "dot" product in Eq. (2.3) there corresponds the respective consequence from the cracovian product in Eq. (3.2). We conclude that there exists a homomorphism between the quasigroups defined by the product in Eq. (2.3), and square cracovians with inverses, provided that the involution operation I in Eq. (2.2) is identified with the operation of transposing a matrix. Consequently, square cracovians with inverses constitute representations of quasigroups connected with groups with involution. The right identity τ is represented by the transposing cracovian T .

Corollary 4.1. To every matrix group with the involution operation, which is the operation

of transposing a matrix, corresponds a cracovian quasigroup.

Definition 4.1. A representation by linear substitutions of a quasigroup QG connected with a group G in Eq. (2.1), is a cracovian quasigroup onto which the quasigroup is homomorphic. It consists of the assignment of a square cracovian $C(\hat{a})$ to each quasigroup element \hat{a} in such a way that,

$$C(\hat{a}) \cdot C(\hat{b}) = C(\hat{a} \cdot \hat{b}), \quad \forall \hat{a}, \hat{b} \in QG \quad (4.1)$$

Observation 4.2. The expression "a reducible set of matrices" is used in the sense of "a completely reducible set of matrices", i. e. to a set of matrices which is equivalent to the direct sum of two or more other sets of matrices. The expression "an irreducible set of matrices" is used in the sense of a set of matrices which is not equivalent to the direct sum of two or more other sets of matrices.

Observation 4.3. The notion of irreducibility of a matrix representation of a group carries over on a cracovian representation of a quasigroup. However, if a set of square tables of complex numbers in its quality of being a cracovian irrep of a quasigroup QG in Eq. (4.1), were identified with the matrices representing the group with involution G , with which the respective quasigroup QG is connected, that set of matrices could be reducible. [21, 22].

Observation 4.4. The notions of faithful or unfaithful matrix representations defined for groups, carry over on cracovian representations of quasigroups connected with groups with involution. Each cracovian quasigroup is its own faithful representation.

Observation 4.5. Two types of cracovian irreps of quasigroups connected with groups with involution are known. (1) There are cracovian irreps of quasigroups connected with groups with involution, which at the same time are matrix irreps of the respective groups with involution. To these belong the single-valued cracovian irreps of the quasigroups connected with the orthogonal and pseudo-orthogonal continuous rotation groups. (2) There are double-valued cracovian irreps of quasigroups connected with groups with involution which are not matrix irreps of those groups at the same time. To these belong the double-valued cracovian irreps of the orthogonal and pseudo-orthogonal rotation groups $SO(3)$, L_+^\uparrow , $SO(3, 2)$, $SO(4, 1)$ [21, 22, 23].

5 The "hidden" symmetry group of Weyl, and the quasigroup connected with a group with involution

It seems that an analogy can be drawn between the "hidden" symmetry group of an object, introduced by Weyl [35], and the quasigroup, connected with a group with involution [22].

Weyl considered a set with a symmetry group G . This can be the set of all roots of a

polynom, the set of space-time points, or of all nodes of a crystal lattice. He showed that the essential features of a set endowed with a structure can be determined by studying the group of all automorphisms $Aut\ G$ of this set, which preserve all structural relations. The group G determines the "obvious" symmetry, and the group $Aut\ G$, the "hidden" symmetry of the set. The concept of a "hidden" symmetry group was discussed from the standpoint of group actions on sets by Florek *et al.*[16], and by Lulek *et al.*[27].

In one of the examples discussed by Weyl, the object is a regular septadecagon. The property under consideration is the possibility of construction of that regular septadecagon with the help of a compass and a ruler. The symmetry group G of the set of vertices of the regular septadecagon is the cyclic group C_{17} . This is the obvious geometric symmetry group of the regular septadecagon. The vertices of a regular septadecagon are determined by the roots of the equation $z^{17} - 1 = 0$, with $z = x + iy$. One of the roots is $z = 1$, and the remaining 16 roots are determined by an algebraic equation of degree 16. The root $z = 1$ determines the starting vertex in the construction of a regular septadecagon. The determination of the positions of the remaining 16 vertices is shown to be connected with the group of permutations of the 16 roots of the algebraic equation of the degree 16. This appears to be the cyclic group C_{16} . Consequently, $Aut\ C_{17} = C_{16}$, which is the hidden symmetry group of a regular septadecagon. The possibility of construction of a regular septadecagon with a compass and a ruler hinges on the group C_{16} .

The group of automorphisms $I(g)$ of the obvious symmetry group G determines the quasigroup QG . The quasigroup QG connected with a group with involution G , could be considered as a hidden symmetry group $Aut\ G$ of the object whose obvious symmetry is determined by the group G . The group of automorphisms defined in Eq. (2.2) is the basis for defining the nonassociative product in Eq.(2.3). This product serves for the definition of a quasigroup which could be recognized as a particular case of Weyl's hidden symmetry group of an object.

6 Conclusions

It has been shown that in groups with involution an automorphism can be defined, which leads to the definition of a nonassociative product. The group elements undergoing this nonassociative product form a certain type of a quasigroup, which has the right unit element but not a left unit element. If in the matrix representation of a group with involution, the operation of involution is identified with the operation of transposition of a matrix, the quasigroup connected with that group has cracovian representations. A cracovian representation is analogous to a matrix representation with two basic differences: The row-by-column product of two matrices is replaced by the column-by-column product of two cracovians, and

there is only the right unit element. The orthogonal and pseudo-orthogonal rotation groups are examples of groups with involution. In the matrix representations of these groups any transposed matrix belongs to the relevant group. The involution operation in the matrix group can therefore be identified with the transposition of a matrix, and we can define the nonassociative product, which can be identified with the cracovian product. An analogy has been drawn between Weyl's "hidden" symmetry group and a quasigroup.

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